

ANALYZING THE WU METRIC ON A CLASS OF EGGS IN \mathbb{C}^n – II

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ABSTRACT. We study the Wu metric for the non-convex domains of the form

$$E_{2m} = \{z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \dots + |z_{n-1}|^2 + |z_n|^2 < 1\},$$

where $0 < m < 1/2$. Explicit expressions for the Kobayashi metric and the Wu metric on such pseudo-eggs E_{2m} are obtained. The Wu metric is then verified to be a continuous Hermitian metric on E_{2m} which is real analytic everywhere except along the complex hypersurface $Z = \{(0, z_2, \dots, z_n) \in E_{2m}\}$. We also show that the holomorphic sectional curvature of the Wu metric for this non-compact family of pseudoconvex domains is bounded above in the sense of currents by a negative constant independent of m . This verifies a conjecture of S. Kobayashi and H. Wu for such E_{2m} .

1. INTRODUCTION

We continue our study of the Wu metric from [1] by focusing on the following class of non-convex pseudo-egg domains

$$(1.1) \quad E_{2m} = \{z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \dots + |z_{n-1}|^2 + |z_n|^2 < 1\},$$

for $0 < m < 1/2$. Such a pseudo-egg cannot be biholomorphically transformed to any bounded convex domain. This follows by comparing the Kobayashi indicatrices, which must be linearly equivalent if the domains are biholomorphic to each other. Indeed, the linear mapping on the tangent space given by the derivative of the biholomorphism renders an equivalence between the Kobayashi indicatrices at the corresponding points. Linear maps preserve convexity and Kobayashi indicatrix of a bounded convex domain is convex. On the other hand, the indicatrix at the origin for a pseudo-egg, (being a copy of E_{2m}), is non-convex. This also means that the Kobayashi metric fails to satisfy the triangle inequality on the tangent space at the origin $T_0 E_{2m}$. While the Wu metric is indeed a norm, it is not clear if the Wu metric enjoys better regularity than the Kobayashi metric. In general, the Wu metric may fail to be upper semicontinuous [8] notwithstanding the fact that the Kobayashi metric is always upper semicontinuous. In fact, in the case of C^2 -smooth convex eggs, i.e. when $m > 1$, the Wu metric is only C^1 -smooth (see [1]) while the Kobayashi metric is C^2 -smooth. For the non-convex pseudo-eggs as in (1.1), first note that ∂E_{2m} is not even C^1 -smooth. Specifically, the non-smooth points of the boundary are given by $\{(z_1, \dots, z_n) \in \partial E_{2m} : z_1 = 0\}$, which is also the boundary of the set $Z = \{(0, z_2, \dots, z_n) \in E_{2m}\}$; remaining piece of the boundary $\partial E_{2m} \setminus Z$ is a smooth strongly pseudoconvex hypersurface. Let us now state our result on the Wu metric for such pseudo eggs.

Theorem 1.1. *For $0 < m < 1/2$, the Wu metric on E_{2m} is a continuous Hermitian metric which is real analytic on E_{2m} except along the thin set Z . It is nowhere-Kähler. Furthermore, its holomorphic sectional curvature is non-constant and is bounded above by a negative constant independent of m , in the sense of currents.*

This result extends the work of Cheung and Kim ([3] and [4]) on pseudo-egg domains in \mathbb{C}^2 . It also verifies the following conjecture of Kobayashi ([10]) (refer [13] for a modified version of this

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conjecture due to Wu) for the pseudo-eggs E_{2m} .

Conjecture K-W: On every Kobayashi complete hyperbolic complex manifold, there exists a C^k -smooth (for some $k \geq 0$) complete Hermitian metric with its holomorphic curvature ¹ bounded above by a negative constant in the sense of currents.

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2. THE WU METRIC ON E_{2m} FOR $0 < m < 1/2$

In order to analyse the Wu metric on E_{2m} , it is natural to first compute the Kobayashi metric on E_{2m} . Note that E_{2m} is a *balanced* pseudoconvex domain in \mathbb{C}^n and hence the Kobayashi metric at the origin, $K_{E_{2m}}((0, 0, \dots, 0), v) = q_{E_{2m}}(v)$ where $q_{E_{2m}}$ denotes the *Minkowski functional* of E_{2m} . Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian inner product in \mathbb{C}^{n-1} , and write $z \in \mathbb{C}^n$ as $z = (z_1, \hat{z})$, where $\hat{z} = (z_2, \dots, z_n)$. For $(p_1, \dots, p_n) \in E_{2m}$ with $p_1 \neq 0$ and Ψ an automorphism of \mathbb{B}^{n-1} that takes \hat{p} to the origin,

$$(2.1) \quad \Phi(z_1, \dots, z_n) = \left(\frac{|p_1|}{p_1} \frac{(1 - |\hat{p}|^2)^{1/2m}}{(1 - \langle \hat{z}, \hat{p} \rangle)^{1/m}} z_1, \Psi(\hat{z}) \right)$$

is an automorphism of E_{2m} . Furthermore,

$$\Phi((p_1, \dots, p_n)) = \left(\frac{|p_1|}{(1 - |p_1|^2)^{1/2m}}, \hat{0} \right).$$

It follows that the study of the Kobayashi and Wu metrics can be focussed on points $(p, \hat{0})$ for $0 \leq p < 1$, as in the case of convex eggs, in view of automorphisms of E_{2m} of the form (2.1).

Theorem 2.1. *For $0 < m < 1/2$, the Kobayashi metric for E_{2m} at the point $(p, \hat{0})$ for $0 < p < 1$ is given by*

$$K_{E_{2m}}((p, \hat{0}), v) = \begin{cases} K_1((p, \hat{0}), v) = \frac{m\alpha(1-t)|v_1|}{p(1-\alpha^2)(m(1-t)+t)} & \text{for } w \leq 1, \\ K_2((p, \hat{0}), v) = \left(\frac{m^2 p^{2m-2} |v_1|^2}{(1-p^{2m})^2} + \frac{|v_2|^2}{1-p^{2m}} + \dots + \frac{|v_n|^2}{1-p^{2m}} \right)^{1/2} & \text{for } w \geq \frac{1}{4m(1-m)}, \\ \min\{K_1((p, \hat{0}), v), K_2((p, \hat{0}), v)\} & \text{for } 1 < w < \frac{1}{4m(1-m)}, \end{cases}$$

where $v = (v_1, \dots, v_n)$ is a tangent vector at the point $(p, \hat{0})$,

$$(2.2) \quad w = \frac{p^2 (|v_2|^2 + \dots + |v_n|^2)}{m^2 |v_1|^2},$$

$$(2.3) \quad t = \frac{2m^2 w}{1 + 2m(m-1)w + (1 + 4m(m-1)w)^{1/2}},$$

and α is the unique positive solution of the equation $\alpha^{2m} - t\alpha^{2m-2} - (1-t)p^{2m} = 0$ in the interval $(0, 1)$. Moreover, there is a $1 < w_0 < \frac{1}{4m(1-m)}$ such that

$$\begin{aligned} K_{E_{2m}}((p, \hat{0}), v) &= K_1((p, \hat{0}), v) & \text{for } w \leq w_0, \\ K_{E_{2m}}((p, \hat{0}), v) &= K_2((p, \hat{0}), v) & \text{for } w \geq w_0. \end{aligned}$$

¹The term ‘holomorphic curvature’ stands precisely for the holomorphic *sectional* curvature and is said to be strongly negative, if it is bounded above by a negative constant.

Furthermore, $w_0 = \frac{t_0}{(m+(1-m)t_0)^2}$ where $t_0 = \frac{x_0^{2m} - p^{2m}}{x_0^{2m-2} - p^{2m}}$ and x_0 is the solution of the equation

$$-(1-m)^2 x^{4m} + (-1-2m+2m^2+p^{2m}) x^{4m-2} - m^2 x^{4m-4} + (1-(2m-1)p^{2m}) x^{2m} + (1+(2m-1)p^{2m}) x^{2m-2} - p^{2m} = 0.$$

The above result extends the computation done for such pseudo-egg domains in \mathbb{C}^2 (Theorem 4 of [12]). The proof relies on the description of complex geodesics with respect to the Kobayashi metric due to Pflug and Zwonek (see Proposition 2 of [12]). The proof of Theorem 2.1 follows using the techniques of [12].

Proposition 2.2. *In terms of the Euclidean coordinates on the tangent bundle $E_{2m} \times \mathbb{C}^n$, for every $(p, \hat{0}) \in E_{2m}$, the unit sphere of the Wu metric in $T_{(p, \hat{0})} E_{2m}$ is given by*

$$r_1 |v_1|^2 + r_2 (|v_2|^2 + \dots + |v_n|^2) = 1$$

where r_1 and r_2 are positive real-valued continuous functions of p .

It turns out that, to determine the best fitting ellipsoid at $(p, \hat{0})$, it suffices to find $r_1, r_2 > 0$ such that the set

$$\{(v_1, \dots, v_n) : v_1 \geq 0, \dots, v_n \geq 0, r_1 v_1^2 + r_2 (v_2^2 + \dots + v_n^2) = 1\}$$

encloses the smallest volume with the coordinate axes and contains the set

$$(2.4) \quad \{(v_1, \dots, v_n) : v_1 \geq 0, \dots, v_n \geq 0, K_{E_{2m}}((p, \hat{0}), v) \leq 1\}.$$

Recall from Theorem 2.1 that there is a $1 < w_0 < \frac{1}{4m(1-m)}$ such that

$$K_{E_{2m}}((p, \hat{0}), v) = \begin{cases} K_1((p, \hat{0}), v) & \text{for } w \leq w_0, \\ K_2((p, \hat{0}), v) & \text{for } w \geq w_0. \end{cases}$$

It follows that the boundary of the set described by equation (2.4) is the union of two curves, determined by whether $w < w_0$ or $w \geq w_0$. Henceforth, the portion of the curve $K_{E_{2m}}^2((p, \hat{0}), v) = 1$ for $w \geq w_0$ and $w < w_0$ will be referred to as the lower K-curve and the upper K-curve respectively. In terms of the square transformation

$$\begin{aligned} x &= v_2^2 + \dots + v_n^2 \text{ and} \\ y &= v_1^2, \end{aligned}$$

the lower K-curve is described by

$$\frac{m^2 p^{2m-2} y}{(1-p^{2m})^2} + \frac{x}{1-p^{2m}} = 1.$$

The upper K-curve is given by the parametric equations, as follows.

$$(2.5a) \quad \begin{cases} x(\alpha) = (v_2(\alpha))^2 + \dots + (v_n(\alpha))^2 = \frac{\alpha^{4m-2} + p^{4m} - p^{2m} \alpha^{2m-2} - p^{2m} \alpha^{2m}}{\alpha^{4m-2}}, \text{ and} \end{cases}$$

$$(2.5b) \quad \begin{cases} y(\alpha) = (v_1(\alpha))^2 = \left(\frac{p(m\alpha^{2m-2} - (m-1)\alpha^{2m} - p^{2m})}{m\alpha^{2m-1}} \right)^2, \end{cases}$$

with α varying between p and x_0 , where x_0 is as in Theorem 2.1. As the upper K-curve can be represented as the graph of a function, we may cast the equations (2.5a) and (2.5b) as a single equation $y(\alpha) = \left(f(\sqrt{x(\alpha)}) \right)^2$. The parametric form for the upper K-curve as above, shows that f is real analytic. It is straightforward to verify that f is strictly square convex for $p < \alpha < x_0$ (see [3]). This fact together with Proposition 2.2 renders the following result.

Proposition 2.3. *The Wu metric on E_{2m} for $m < 1/2$ at the point $(p, \hat{0})$, $0 < p < 1$ is given by*

$$h_{E_{2m}}(p, \hat{0}) = \frac{1}{(1-p^2)^2} dz_1 \otimes d\bar{z}_1 + \frac{1}{(1-p^{2m})} dz_2 \otimes d\bar{z}_2 + \dots + \frac{1}{(1-p^{2m})} dz_n \otimes d\bar{z}_n.$$

Further, using the invariance of the Wu metric under automorphisms of E_{2m} (as described in (2.1)), we arrive at the following result.

Theorem 2.4. *For $0 < m < 1/2$, the Wu metric on E_{2m} at the point (z_1, \dots, z_n) is given by*

$$\sum_{i,j=1}^n h_{i\bar{j}}(z_1, \dots, z_n) dz_i \otimes d\bar{z}_j,$$

where

$$\begin{aligned} h_{1\bar{1}}(z_1, \dots, z_n) &= \frac{(1 - |\hat{z}|^2)^{1/m}}{\left((1 - |\hat{z}|^2)^{1/m} - |z_1|^2\right)^2}, \\ h_{1\bar{j}}(z_1, \dots, z_n) &= \frac{(1 - |\hat{z}|^2)^{-1+1/m} \bar{z}_1 z_j}{m \left((1 - |\hat{z}|^2)^{1/m} - |z_1|^2\right)^2} \text{ for } 2 \leq j \leq n, \\ h_{i\bar{1}}(z_1, \dots, z_n) &= \overline{h_{1\bar{i}}}(z_1, z_2, \dots, z_n) \text{ for } 2 \leq i \leq n, \\ h_{j\bar{j}}(z_1, \dots, z_n) &= \left(\frac{(1 - |\hat{z}|^2)^{-2+1/m} |z_1|^2 |z_j|^2}{m^2 \left((1 - |\hat{z}|^2)^{1/m} - |z_1|^2\right)^2} + \frac{1 - |\hat{z}|^2 + |z_j|^2}{(1 - |\hat{z}|^2)(1 - |\hat{z}|^2 - |z_1|^{2m})} \right) \text{ for } 2 \leq j \leq n, \\ h_{i\bar{j}}(z_1, \dots, z_n) &= \left(\frac{(1 - |\hat{z}|^2)^{-2+1/m} |z_1|^2 \bar{z}_i z_j}{m^2 \left((1 - |\hat{z}|^2)^{1/m} - |z_1|^2\right)^2} + \frac{\bar{z}_i z_j}{(1 - |\hat{z}|^2)(1 - |\hat{z}|^2 - |z_1|^{2m})} \right) \text{ for } 2 \leq i, j \leq n \end{aligned}$$

and $i \neq j$.

It follows that the Wu metric is real analytic on the set $\{(z_1, \dots, z_n) \in E_{2m} : z_1 \neq 0\}$. Moreover, the Wu metric is continuous at points $(0, z_2, \dots, z_n)$ of E_{2m} . Indeed, the continuity and completeness of the Wu metric on E_{2m} follows from Proposition 4 of [8]. Completeness of the Wu metric here relies on Kobayashi completeness of the domains E_{2m} , which is guaranteed by [11]. Furthermore, it is straightforward to see that

$$\frac{\partial h_{12}}{\partial z_2}(z_1, \hat{0}) \neq \frac{\partial h_{22}}{\partial z_1}(z_1, \hat{0}).$$

This shows (see, for instance, Lemma 3.2 of [6]) that the Wu metric is not Kähler at these reference points. Since $E_{2m} \setminus Z$ is the orbit of $\{(z_1, \hat{0}) \in E_{2m} : z_1 \neq 0\}$ under the action of the automorphism group of E_{2m} , it follows that the Wu metric is not Kähler on $E_{2m} \setminus Z$. Notice that $E_{2m} \setminus Z$ is an open dense subset of E_{2m} and hence, the following result.

Corollary 2.5. *For any $0 < m < 1/2$, the Wu metric on E_{2m} is nowhere Kähler.*

3. NEGATIVE HOLOMORPHIC SECTIONAL CURVATURE

Let $G = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ be a C^2 -smooth Hermitian metric on a complex manifold X . Then the holomorphic sectional curvature of G along the direction of $\xi = (\xi_1, \dots, \xi_n) \in T_p X$ at $p \in X$

is given by

$$\frac{\sum R_{i\bar{j}k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l}{\sum g_{i\bar{j}}(p)g_{k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l},$$

where $R_{i\bar{j}k\bar{l}}$ are the components of the curvature tensor given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{\alpha, \beta} g^{\alpha\bar{\beta}} \frac{\partial g_{i\bar{\beta}}}{\partial z_k} \frac{\partial g_{\alpha\bar{j}}}{\partial \bar{z}_l}.$$

Here, $(g^{\alpha\bar{\beta}})$ denotes the inverse of the matrix $(g_{\alpha\bar{\beta}})$.

In case G is only continuous (and not C^2 -smooth), then the holomorphic sectional curvature is defined in a distributional sense, as a *current* of type $(1, 1)$ (cf. following [13]) and is said to be bounded above by a negative constant c if every embedded Riemann surface S in X with $H|_S = h_0 d\xi \otimes d\bar{\xi}$ satisfies

$$\Delta_\xi \log h_0 = \frac{\partial^2 \log h_0}{\partial \xi \partial \bar{\xi}} > -ch_0 \partial \xi \wedge \partial \bar{\xi},$$

in the sense of currents.

Proposition 3.1. *The holomorphic sectional curvature of the Wu metric on E_{2m} , $0 < m < 1/2$, is bounded above by $-1/2$ at every point where the Wu metric is smooth.*

Proof. A direct computation using Theorem 2.4 shows that for $0 < p < 1$,

$$\begin{aligned} R_{1\bar{1}1\bar{1}}((p, \hat{0})) &= -\frac{2}{(1-p^2)^4}, \\ R_{1\bar{1}j\bar{j}}((p, \hat{0})) &= -\frac{1+p^2}{m(1-p^2)^3} + \frac{p^2(1-p^{2m})}{m^2(1-p^2)^4} \text{ for } 2 \leq j \leq n, \\ R_{1\bar{i}i\bar{i}}((p, \hat{0})) &= R_{i\bar{1}i\bar{i}}((p, \hat{0})) = -\frac{1+p^2}{m(1-p^2)^3} + \frac{p^{2m}}{(1-p^2)^2(1-p^{2m})} \text{ for } 2 \leq i \leq n, \\ R_{i\bar{i}1\bar{1}}((p, \hat{0})) &= -\frac{m^2 p^{2m-2}}{(1-p^{2m})^3} \text{ for } 2 \leq i \leq n, \\ R_{i\bar{i}j\bar{j}}((p, \hat{0})) &= -\frac{1}{(1-p^{2m})^2} \text{ for } 2 \leq i, j \leq n \text{ and } i \neq j, \\ R_{i\bar{j}j\bar{i}}((p, \hat{0})) &= -\frac{p^2}{m^2(1-p^2)^2} - \frac{1}{1-p^{2m}} \text{ for } 2 \leq i, j \leq n \text{ and } i \neq j, \\ R_{j\bar{j}j\bar{j}}((p, \hat{0})) &= -\frac{p^2}{m^2(1-p^2)^2} - \frac{1}{1-p^{2m}} - \frac{1}{(1-p^{2m})^2} \text{ for } 2 \leq j \leq n, \end{aligned}$$

and all other curvature components vanish. Now, arguments similar to those employed in [3] using the above computations yield that

$$\sum R_{i\bar{j}k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l < -\frac{1}{2} \sum g_{i\bar{j}}(p)g_{k\bar{l}}(p)\xi_i\bar{\xi}_j\xi_k\bar{\xi}_l.$$

□

It remains to establish the uniform negativity of curvature on the thin set Z .

Proposition 3.2. *There is a negative constant c such that the holomorphic sectional curvature of the Wu metric at points of $Z \subset E_{2m}$, is bounded above by c in the sense of currents. Moreover, the constant c is independent of m .*

To prove this proposition, one cannot rely solely on the arguments used in the case of convex eggs (i.e., those presented in Appendix B of [3]), since the Wu metric is not even C^1 -smooth on Z . However, the pseudo-eggs have the following property – for $m < 1/2$, $E_{2m} \subset \mathbb{B}^n$, where \mathbb{B}^n denotes the unit ball in \mathbb{C}^n . Hence, one can pull back the standard Poincare-Bergman metric g on \mathbb{B}^n by the inclusion mapping $i : E_{2m} \rightarrow \mathbb{B}^n$, to get the metric i^*g on E_{2m} and compare it with the Wu metric $h_{E_{2m}}$. It follows that $i^*g \leq \sqrt{n}h_{E_{2m}}$. Moreover, in this case, the best fitting ellipsoid at the origin is \mathbb{B}^n . Furthermore, since the Kobayashi-indicatrix at the origin is (a copy of) E_{2m} , it follows that i^*g coincides with $h_{E_{2m}}$ at the origin. Now, let S be any embedded Riemann surface with complex coordinate s and passing through the origin at $s = 0$. To establish the strong negativity of the holomorphic curvature near the origin, in the sense of currents, compare the restriction $G = i^*g|_S$ with $H = h_{E_{2m}}|_S$. Real analyticity of G ensures that its logarithmic average on small ‘discs’ about $s = 0$ in S equals $\Delta \log G(s)$ at $s = 0$ – this can be verified by a Taylor expansion of G in s, \bar{s} . The decreasing property of holomorphic curvature together with the facts that G is a conformal metric on S of constant negative holomorphic curvature and $H \geq 1/\sqrt{n}G$, establishes the strong negativity of the holomorphic curvature current of the Wu metric in a neighbourhood of the origin as in [4]. Notice that the bound on the curvature that we get here, is a constant that depends on the dimension n . But this constant does not depend on m . As Z is the orbit of the origin under the action of $\text{Aut}(E_{2m})$, this analysis carries forth to hold throughout Z .

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